Alternative Boundaries for CUSUM Tests

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Abstract

Alternative boundaries for the common Recursive (or Standard) CUSUM test and the OLS-based CUSUM test for structural change are suggested and their properties are examined by simulation of expected p values. The poor power of the tests for early and late structural changes can be improved for the OLS-based version, whereas this weakness of the Recursive CUSUM test cannot be overcome by the new boundaries.

Keywords: CUSUM test, structural change.

1. Introduction and summary

We consider two well known tests for structural change in the fluctuation test framework (Kuan and Hornik 1995): the CUSUM tests either based on recursive residuals as suggested by Brown, Durbin, and Evans (1975) or on the usual OLS residuals as introduced by Ploberger and Krämer (1992). Both are suitable for testing the constancy of regression coefficients in linear regression relationships, but are known to have poor power against early and late structural changes.

To spread the rejection probability under the null hypothesis more evenly, Brown *et al.* (1975) suggest boundaries that are proportional to the standard deviation of the limiting process. But they confine themselves to linear boundaries because for these the theoretical crossing probabilities are known, from e.g. Durbin (1971). Krämer and Sonnberger (1986) also argue that it would be desirable to use these alternative boundaries. Therefore we simulate critical values for such boundaries and compare the resulting tests by simulation of expected p values. Although an improvement cannot be accomplished for the Recursive CUSUM test due to the properties of the recursive residuals under the alternative, the rejection probabilities of the OLS-based CUSUM test with alternative boundaries are indeed rather evenly distributed for structural changes early, midway and late in the sample period.

2. The model and the tests

Consider the standard linear regression model

$$y_i = x_i^\top \beta_i + u_i \qquad (i = 1, \dots, n), \tag{1}$$

where at time i, y_i is the observation of the dependent variable, x_i is a $k \times 1$ vector of regressor variables, with the first component equal to unity, β_i is the $k \times 1$ vector of regression coefficients and u_i is an error term. CUSUM tests are concerned with testing the hypothesis $H_0 : \beta_i = \beta_0$ for $i = 1, \dots, n$ against the alternative that β_i varies over time. Like Ploberger and Krämer (1992) and Krämer, Ploberger, and Alt (1988) we assume that the regressors x_i and the disturbances u_i are defined on a common probability space, such that $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n ||x_i||^{2+\delta} < \infty$ a.s. for some $\delta > 0$ ($|| \cdot ||$ the Euclidean norm), and that $\frac{1}{n} \sum_{i=1}^n x_i x_i^\top \xrightarrow{p} Q$ for some finite regular matrix Q. Furthermore it is assumed that the disturbances u_i are stationary and ergodic, with $\mathrm{E}[u_i|\mathcal{A}_i] = 0$ and $\mathrm{E}[u_i^2|\mathcal{A}_i] = \sigma^2$, where $\mathcal{A}_i$ is the σ-field generated by $\{y_s, x_s, u_s | s < i\}$. These

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assumptions allow in particular for dynamic models, in which case they imply stability.

The Recursive CUSUM test is based on the cumulative sum of the recursive residuals

$$\tilde{u}_i \;=\; \frac{y_i - x_i^\top \hat{\beta}^{(i-1)}}{\sqrt{1 + x_i^\top \left(X^{(i-1)\top} X^{(i-1)} \right)^{-1} x_i}} \qquad (i = k+1, \dots, n), \tag{2}$$

which have zero mean and variance σ^2 under the null hypothesis. $\hat{\beta}^{(i-1)}$ is the ordinary least squares estimate of the regression coefficients based on the observations up to $i-1$. The Recursive CUSUM process is defined as

$$W_n(t) \;=\; \frac{1}{\tilde{\sigma}\sqrt{n-k}} \sum_{i=k+1}^{\lfloor k+t(n-k) \rfloor} \tilde{u}_i \qquad (0 \le t \le 1), \tag{3}$$

where $\tilde{\sigma} = \sqrt{\frac{1}{n-k}\sum_{i=k+1}^{n}(\tilde{u}_i - \bar{\tilde{u}})^2}$. Krämer *et al.* (1988) show that for $n \to \infty$, $W_n \Rightarrow W$, where $\Rightarrow$ denotes convergence in distribution and $W(t)$ is the standard Wiener process (or Brownian motion).

If there is just a single structural change at time $t_0 < 1$ the mean of the recursive residuals will be zero only up to t_0 and different afterwards. Hence the CUSUM path $W_n(t)$ will start to leave its zero mean at t_0. H_0 is rejected whenever $W_n(t)$ crosses either of the classical linear boundaries $b(t) = \lambda \cdot (1 + 2t)$ or $-b(t)$, where λ depends on the significance level α of the test.

The OLS-based CUSUM test is defined analogously using the OLS residuals $\hat{u}_i = y_i - x_i^\top \hat{\beta}$. The OLS-based CUSUM process is defined as

$$W_n^0(t) \;=\; \frac{1}{\hat{\sigma}\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \hat{u}_i \qquad (0 \le t \le 1), \tag{4}$$

where $\hat{\sigma} = \sqrt{\frac{1}{n-k}\sum_{i=1}^{n}\hat{u}_i^2}$. This path will always not only start in zero but also return to zero, but if there is a single structural change at t_0 it should have a peak close to the break point t_0. H_0 is rejected if the path crosses either $b_0(t) = \lambda$ or $-b_0(t)$. This corresponds to rejection if the maximum of the process is too large, alternatively Krämer and Schotman (1992) suggest to use the range of the process as a test statistic. Ploberger and Krämer (1992) show that for $n \to \infty$ the process $W_n^0 \Rightarrow W^0$, where $W^0(t) = W(t) - tW(1)$ is the standard Brownian bridge.

3. Alternative boundaries

One of the major drawbacks of both CUSUM tests is their poor power for early and late structural changes. To have similar properties over the whole time interval it seems natural to consider boundaries that are proportional to the standard deviation of the limiting process, so that the rejection probability under H_0 is spread evenly. Thus the alternative boundary for the Recursive CUSUM process is $\tilde{b}(t) = \lambda \cdot \sqrt{t}$, as the variance of a Standard Brownian motion is $\mathrm{VAR}[W(t)] = t$. Brown *et al.* (1975) already suggested this boundary, because they expected more evenly distributed rejection properties from it. But they used the boundary $b(t)$ instead (which is tangential to $\tilde{b}(t)$ in $t = 0.5$) because there is a closed form solution for the crossing probability of the Wiener process. Similarly, the alternative boundary for the OLS-based CUSUM process is $\tilde{b}_0(t) = \lambda \cdot \sqrt{t(1-t)}$ as $\mathrm{VAR}[W^0(t)] = t(1-t)$.

The edges of the interval [0,1] have to be treated with some care because for $t = 0$ the rejection condition $W(0) \ge \tilde{b}(0) = 0$ would be satisfied trivially. Even on the interval (0,1] the rejection probability would converge to 1, so a compact interval $[\varepsilon, 1]$ with $\varepsilon > 0$ is needed, here $\varepsilon = 0.001$ is used. For the OLS-based CUSUM test both ends of the interval have to be excluded, so actually $[\varepsilon, 1-\varepsilon]$ is used.

α	Recursive CUSUM	OLS-based CUSUM
0.10	2.90	3.13
0.05	3.15	3.37
0.01	3.65	3.83

Table 1: Critical values for the CUSUM tests with alternative boundaries

The critical values for the boundaries $\tilde{b}(t)$ and $\tilde{b}_0(t)$ are assessed via the algorithm suggested by Wang and Pötzelberger (1997) and refined by Pötzelberger and Wang (2001) for crossing probabilites of Brownian motions for arbitrary boundaries: The boundaries are approximated by piecewise linear functions by simple interpolation in 128 sub-intervals of identical size. The formula provided by Wang and Pötzelberger (1997) is evaluated 200,000 times yielding an estimate of half of the crossing probability, because only one-sided crossing probabilities are considered. The probability that a Brownian motion crosses both boundaries is neglegible if the crossing probability is small (e.g. for significance levels 1%, 5%, 10%). Furthermore the algorithm provides an estimate of the standard deviation, which was smaller than 10^{-6} for all simulated values. The critical values for the common significance levels are given in Table 1. All computations were carried out in R (`http://www.R-project.org/`), in particular using the package `strucchange` (Zeileis, Leisch, Hornik, and Kleiber 2002).

To compare the shape of the linear and the alternative rejection area, both boundaries for the significance level $\alpha = 0.01$ are plotted in Figure 1. The shape for other significance levels is similar. It can be seen that the alternative boundary for the Recursive CUSUM process offers advantages only for $t \leq 0.2$. Therefore structural changes that occur late in the sample can be detected more easily with the linear boundaries; even for early shifts there is little hope that the advantage of the alternative boundary can be used as Figure 2 illustrates: Although the break point is at 10% of the 1000 observations in this simulated data set, the CUSUM path actually crosses the linear boundary much later and fails to cross the alternative boundary.

In contrast to this the alternative boundary for the OLS-based CUSUM process lies under the linear one at the beginning as well as at the end, as seen on the right of Figure 1. These advantages

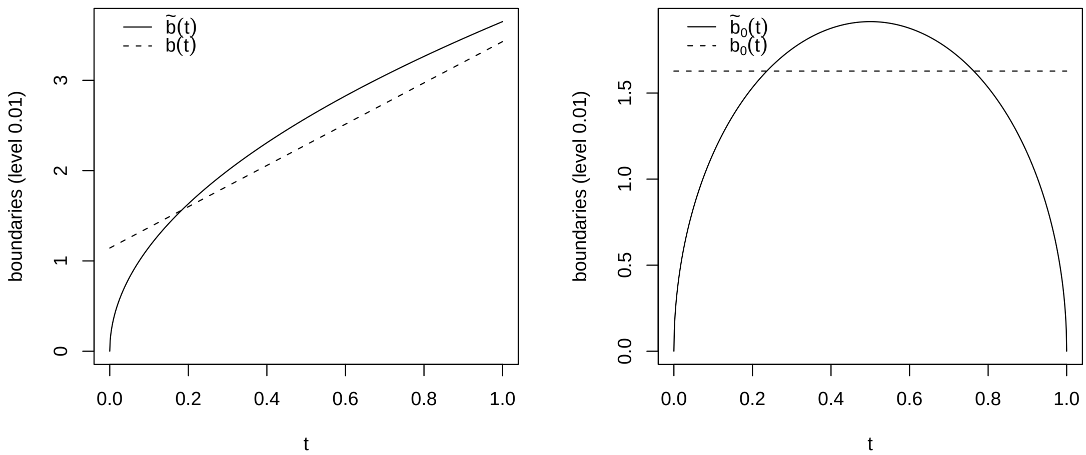

Figure 1: Comparison of the linear and alternative boundaries for the Recursive and the OLS-based CUSUM test

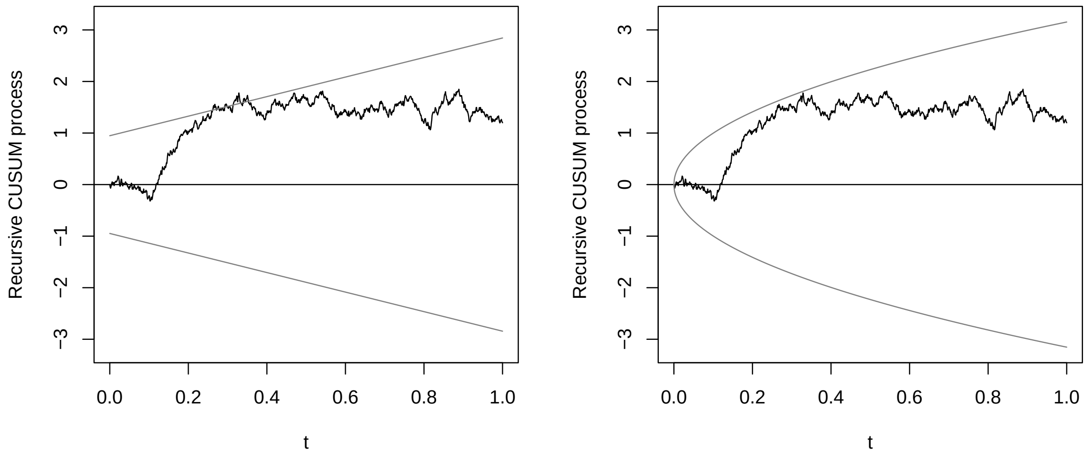

Figure 2: Comparison of the Recursive CUSUM tests

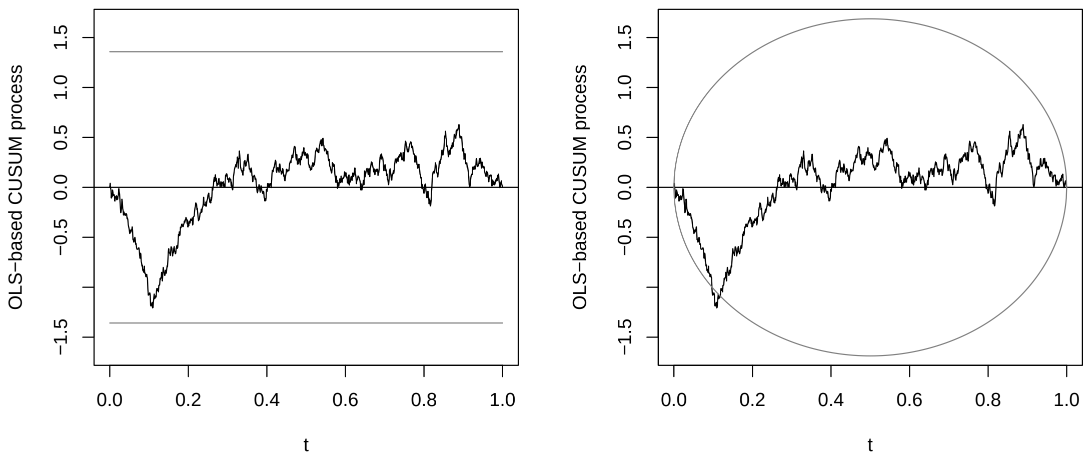

Figure 3: Comparison of the OLS-based CUSUM tests

can be worth the disadvantage in the middle as Figure 3 indicates, which shows the OLS-based CUSUM process for the same simulated data as above with a structural shift at $t = 0.1$.

Whereas the linear boundaries fail to detect the structural change at level $\alpha = 0.01$, the new boundaries are able to find evidence for a structural shift at the same level. The reason is the behaviour of the OLS-based CUSUM process under the alternative: the path has its peak around the break point so that the advantages of the alternative boundaries can be used for early and late structural changes. To emphasize this, expected p values will be simulated in the next section.

4. Simulation of expected p values

If T_0 is the test statistic distributed according to the null distribution F_0 and T—the test statistic under some specified alternative F_θ—takes the value t, the usual p value is given by $\mathrm{P}(T_0 \geq t | T = t)$. Thus the expected p value (EPV) according to Sackrowitz and Samuel-Cahn (1999) is the unconditional probability $EPV(\theta) = \mathrm{P}(T_0 \geq T)$, which is $1 -$ expected power (over all possible levels). Under H_0 the expected p value is obviously 0.5; a small expected p value indicates good chances to reject H_0.

Expected p values seem to be convenient for comparing the power of the two OLS-based CUSUM tests as they do not depend on the significance level; the power of the two Recursive CUSUM tests will not be compared because the alternative boundaries did not offer any advantages.

A simple model for local single shift alternatives is chosen as in Ploberger and Krämer (1992) with $k = 2$, $x_i = (1, (-1))^\top$ and $u_i \sim \mathrm{nid}(0,1)$. We use time series of length $n = 500$. Then the timing q, intensity d and angle ψ of a single shift are varied in the following way:

$$\beta_i \quad = \quad \left\{ \begin{array}{rl} \beta & \text{for } t \leq \lfloor qn \rfloor \\ \beta + \Delta\beta & \text{for } t > \lfloor qn \rfloor \end{array} \right. , \tag{5}$$

and the shift is given by $\Delta\beta = n^{-0.5} d (\cos\psi, \sin\psi)^\top$, where ψ is the angle between the shift and the mean regressor $(1, 0)^\top$. Including the angle is necessary as neither the Recursive nor the OLS-based CUSUM test are able to pick up shifts with an angle of 90° (Ploberger and Krämer 1992). The intensity of the shift is $||\Delta\beta|| = |d|\sqrt{n}$, which occurs at time $t = \lfloor qn \rfloor$ with q taking values 0.1, 0.5 and 0.9. So structural changes early, midway and late in the sample period are covered. In 10,000 runs one test statistic under H_0 and one under the specified alternative are simulated and it is checked whether the null test statistic is larger. The empirical probabilities are reported in Table 2 and it can be seen that the linear boundaries cause some weaknesses for early and late structural changes, whereas the properties of the test are rather good in the middle. The alternative boundaries can overcome the weakness for early and late changes and they spread the rejection probability more evenly over the whole sample period.

Figure 4 shows a dotplot in which the simulated EPVs are plotted against the angle for each combination of intensity and timing, stratified by the boundary type. It can be seen that the EPV decreases with intensity and approaches 0.5 as the angle increases. When comparing the boundary types, Figure 4 reveals that the EPV of the alternative boundaries is usually smaller for early and late changes ($q = 0.1, 0.9$) and greater for $q = 0.5$. Furthermore the advantages of the alternative boundaries are very clear for stronger shifts ($d = 9.6, 12$), whereas the disadvantages almost vanish. Hence, early and late shifts can be detected more easily with the alternative boundaries and the power is comparable in the middle of the sampling period.

		Linear boundaries ψ					Alternative boundaries ψ				
q	d	0°	18°	36°	54°	72°	0°	18°	36°	54°	72°
0.1	4.8	0.340	0.343	0.383	0.406	0.454	0.312	0.326	0.363	0.402	0.448
	7.2	0.250	0.261	0.304	0.359	0.432	0.203	0.219	0.267	0.333	0.434
	9.6	0.180	0.185	0.235	0.305	0.407	0.122	0.146	0.191	0.275	0.398
	12	0.113	0.126	0.177	0.254	0.379	0.062	0.073	0.121	0.222	0.372
0.5	4.8	0.132	0.148	0.186	0.268	0.373	0.165	0.188	0.243	0.318	0.402
	7.2	0.047	0.050	0.090	0.171	0.322	0.061	0.073	0.114	0.213	0.368
	9.6	0.012	0.015	0.033	0.096	0.262	0.014	0.019	0.048	0.126	0.316
	12	0.002	0.003	0.010	0.055	0.208	0.002	0.004	0.017	0.070	0.263
0.9	4.8	0.343	0.353	0.369	0.412	0.452	0.304	0.323	0.354	0.395	0.465
	7.2	0.262	0.267	0.306	0.364	0.435	0.206	0.220	0.267	0.335	0.430
	9.6	0.180	0.193	0.239	0.320	0.408	0.125	0.136	0.186	0.270	0.408
	12	0.120	0.125	0.176	0.264	0.383	0.062	0.071	0.118	0.220	0.375

Table 2: Simulated expected p values of the OLS-based CUSUM test

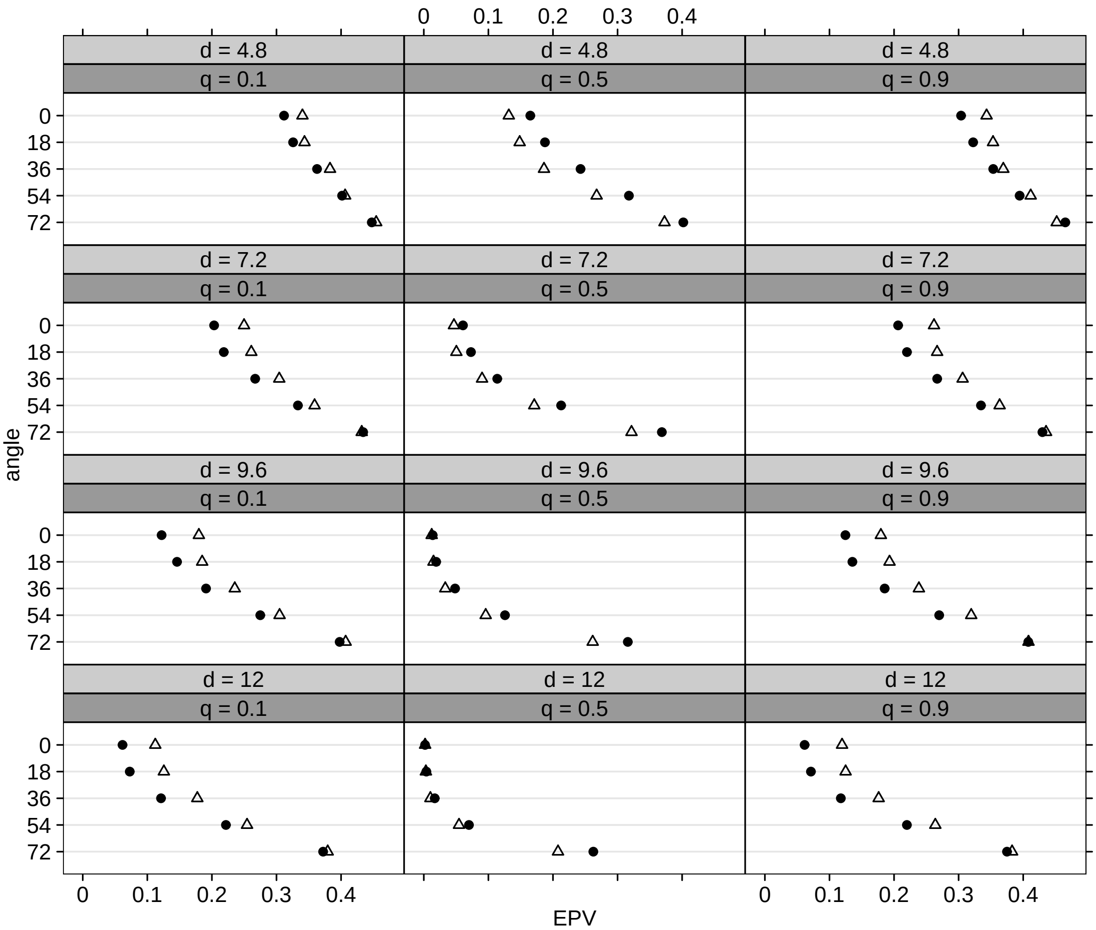

Figure 4: Dotplot of expected p values ($\triangle$ linear and $\bullet$ alternative boundary)

5. Conclusions

Alternative boundaries for the Recursive and the OLS-based CUSUM test which are proportional to the standard deviation of the respective limiting processes are suggested. They fail to improve on the properties of the Recursive CUSUM test, but they can overcome the weakness of the OLS-based CUSUM test for early and late structural changes. If a CUSUM test should be applied to data where the potential break point is not known, the alternative OLS-based CUSUM test is therefore the most recommendable.

The problem of determining "optimal" boundaries for certain types of alternatives deserves further study.

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