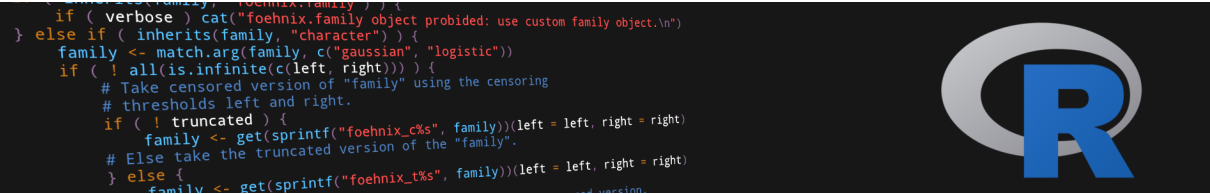
universität innsbruck

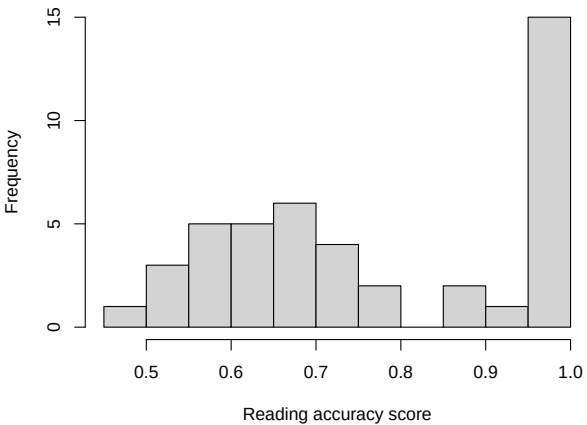


Regression Models for $[0, 1]$ Responses Using betareg and crch

Ioannis Kosmidis, Achim Zeileis

https://www.zeileis.org/

Motivation



Goal: Model limited response variables in unit interval.

Examples: Fractions or proportions (not from independent Bernoulli trials).

Illustration: Reading `accuracy` of 44 primary school children, explained by `dyslexia` status and `iq` score.

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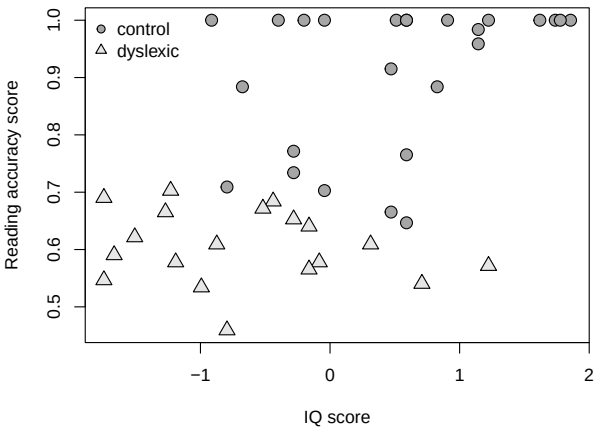

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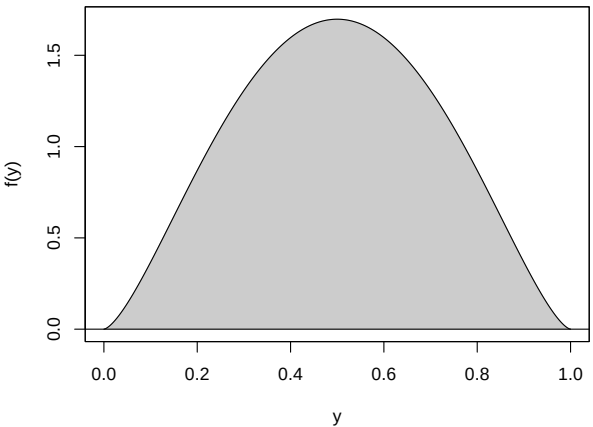
Beta distribution



Parameters: Mean μ, precision ϕ.

Regression: Link both parameters to predictors.

Advantage: Flexible shape, full likelihood.

Disadvantage: Zero probability for 0 and 1.

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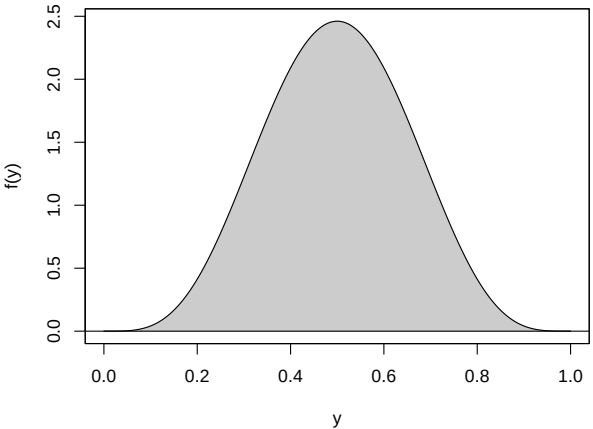

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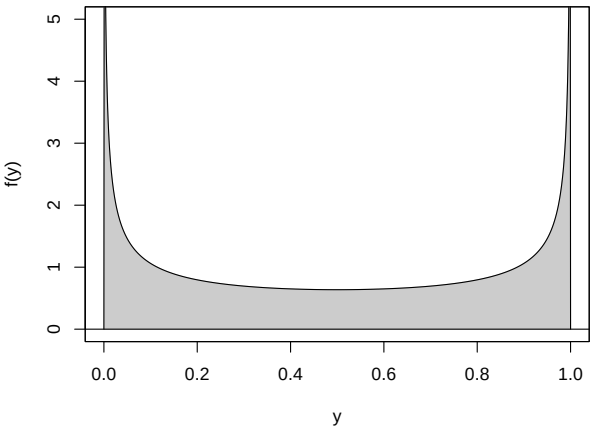

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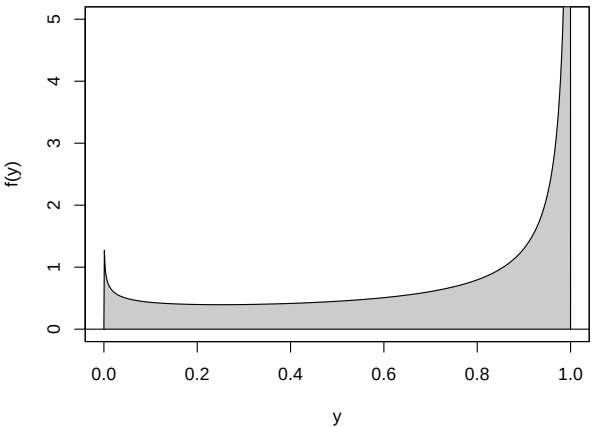

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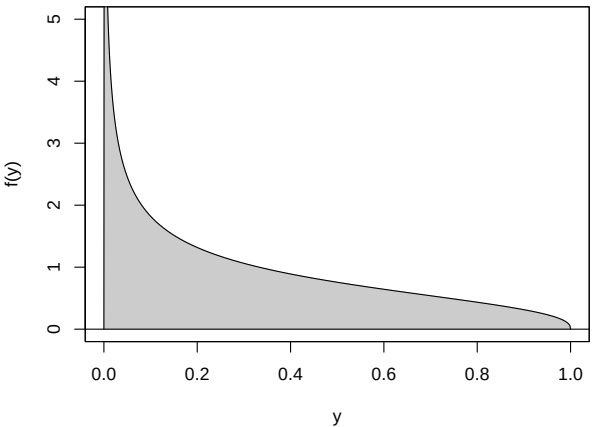

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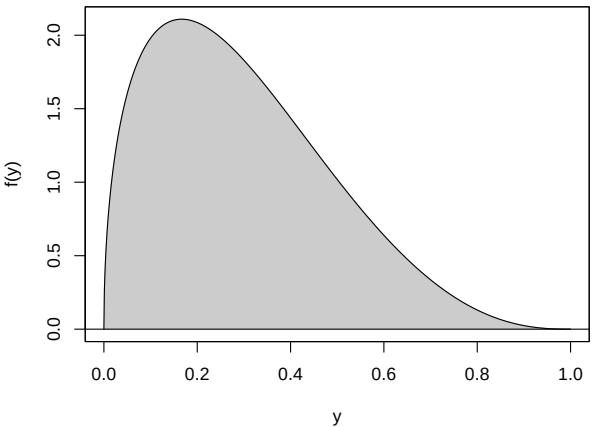

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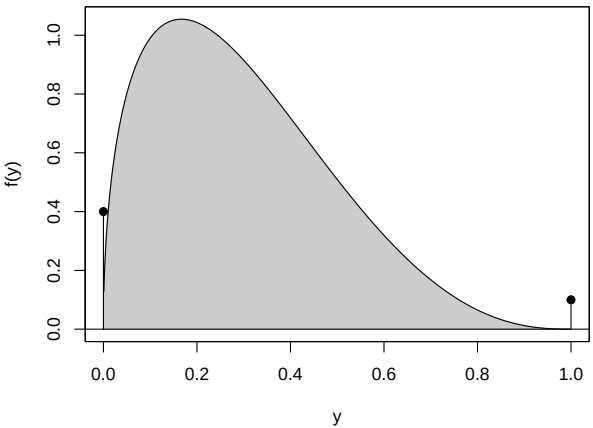
Zero-and/or-one-inflated beta distribution



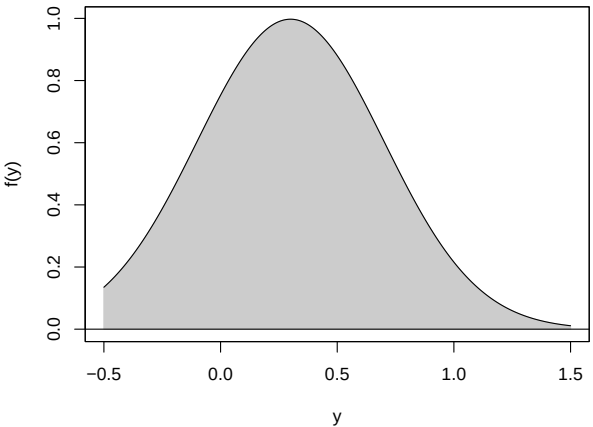
Parameters: Mean μ, precision ϕ, point masses π_0, π_1.

Regression: Link all four parameters to predictors.

Advantage: Keep flexibility, accomodate boundaries.

Disadvantage: Many parameters, separate determinants for boundaries.

Censored normal distribution (tobit)



Parameters: Mean μ, variance σ^2.

Regression: Link both parameters to predictors.

Advantage: No additional determinants for boundaries.

Disadvantage: Less flexible than beta.

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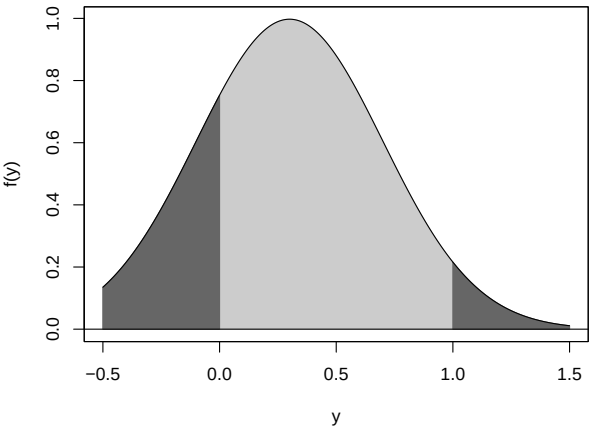

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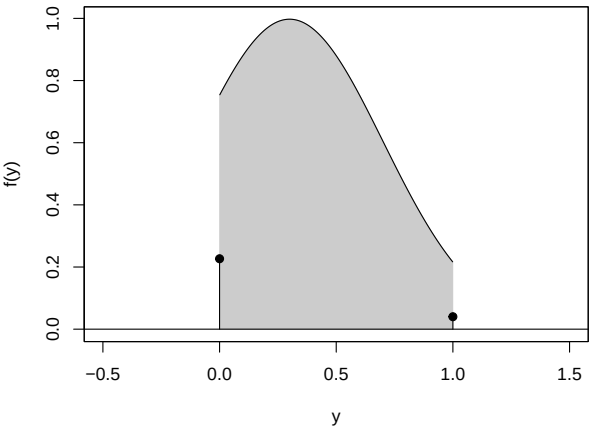

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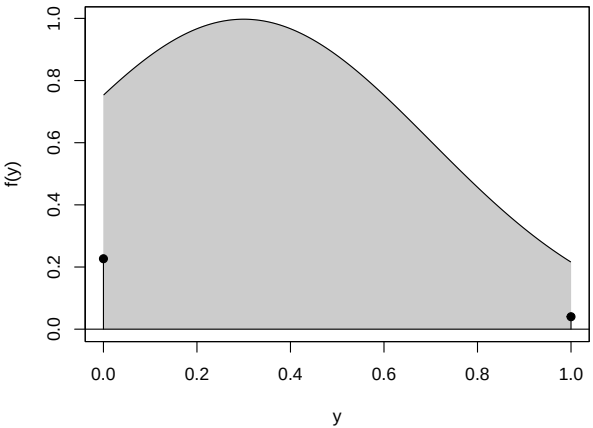

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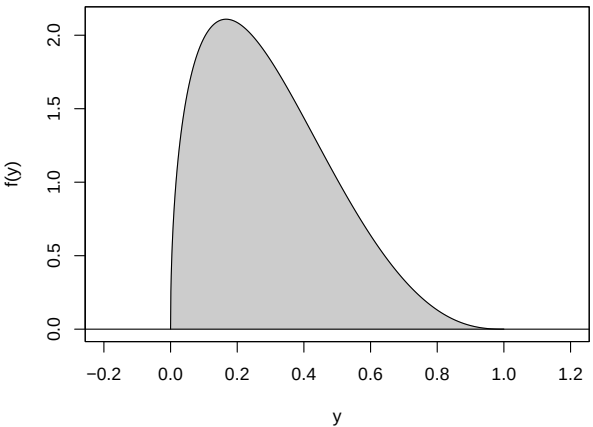
Extended-support beta mixture distribution



Parameters: Mean μ, precision ϕ, exceedence ν.

Regression: Link only μ and ϕ to predictors.

Advantage: Single parameter ν links normal and beta.

Disadvantage: Somewhat more complex.

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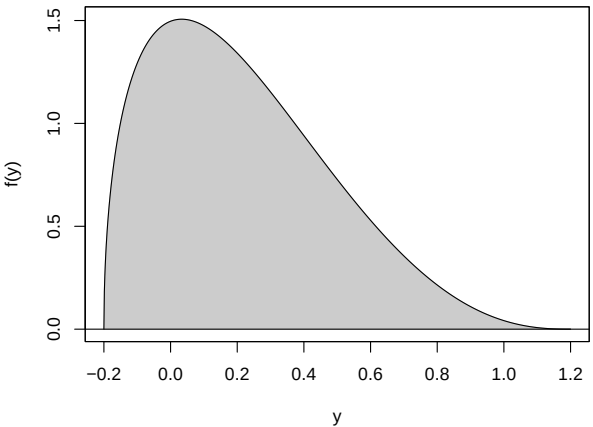

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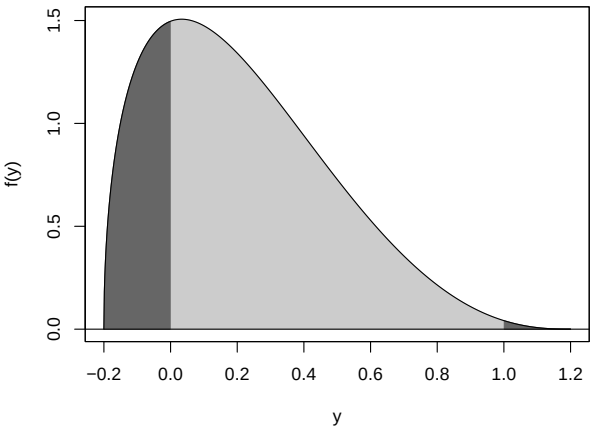

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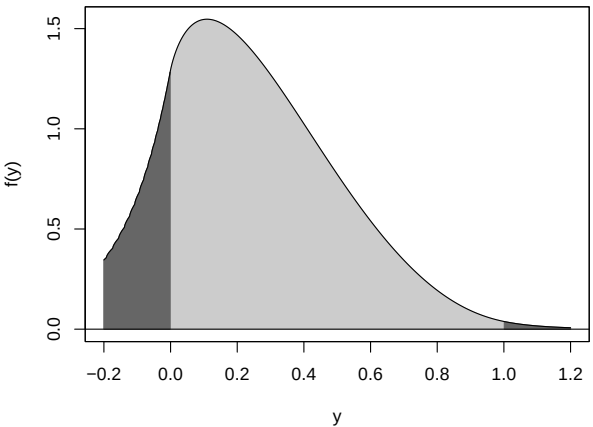

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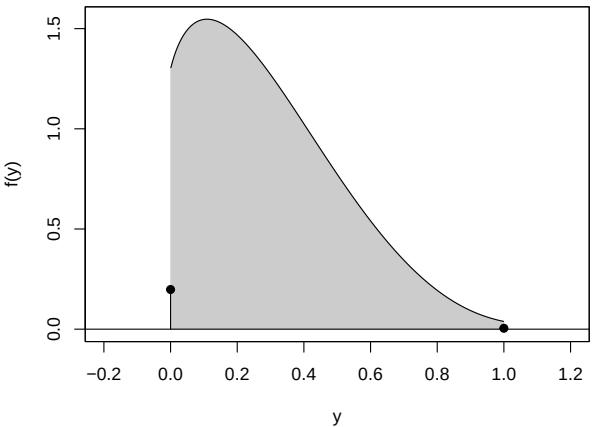

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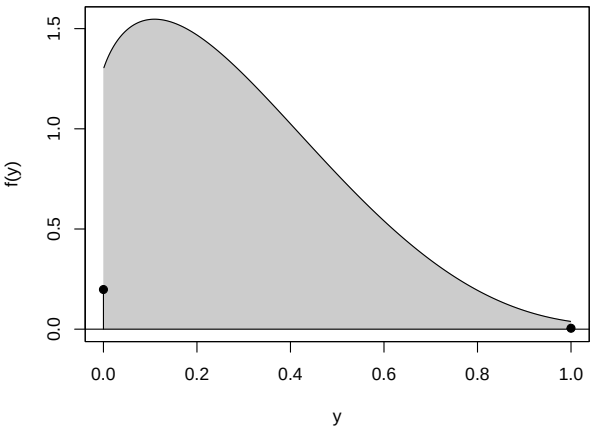

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Reading skills

R packages:

- *crch* for censored (and truncated) regression with conditional heteroscedasticity (default: normal).
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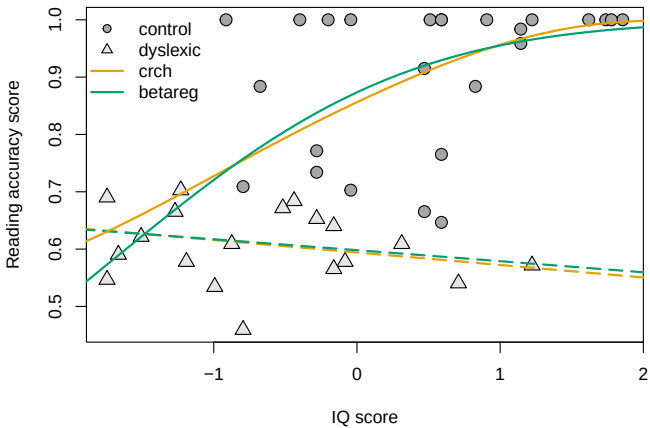
Application: Reading skills data.

```
R> rs1 <- crch(accuracy1 ~ dyslexia * iq | dyslexia + iq, data = ReadingSkills,
+    left = 0, right = 1)
R> rs2 <- betareg(accuracy1 ~ dyslexia * iq | dyslexia + iq, data = ReadingSkills)
```

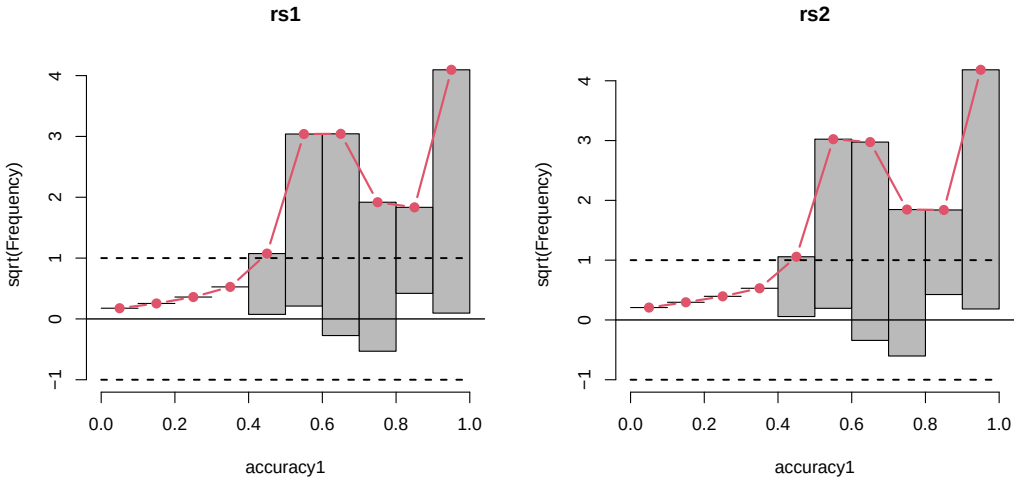
Reading skills

Comparison: Similar effects for expectations.



Reading skills

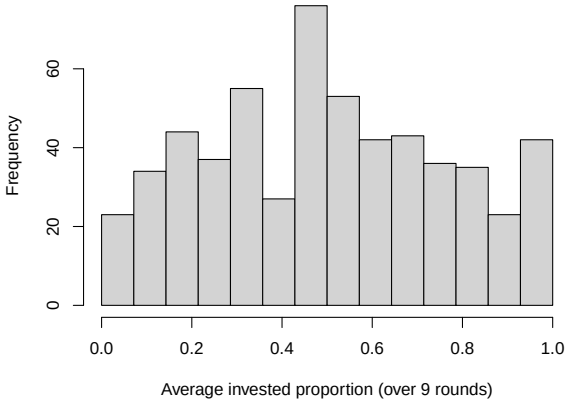
Comparison: Similar fitted distributions in hanging rootograms.



Loss aversion

Behavioral economics experiment: Glätzle-Rützler *at al.* (2015).

- Determinants of loss aversion in high-school students.
- Proportion of tokens invested in risky lottery with positive expected payouts.



Loss aversion

Original analysis: Normal linear regression model with `grade` (lower vs. upper), `arrangement` (single vs. team of two), `male` (at least one), `age`.

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R> la_ols <- glm(invest ~ grade * (arrangement + age) + male, data = LossAversion)
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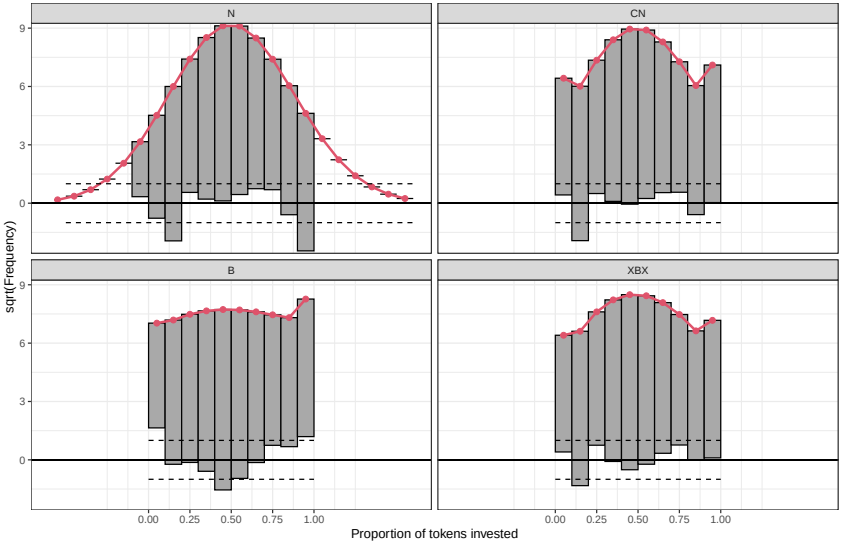
Alternatively: Probabilistic models to simultaneously model expected investments *and* probability to behave like a rational *homo oeconomicus*.

- CN: Heteroscedastic censored normal model.
- B: Beta regression after ad-hoc scaling to the open unit interval.
- XBX: Extended-support beta mixture model.

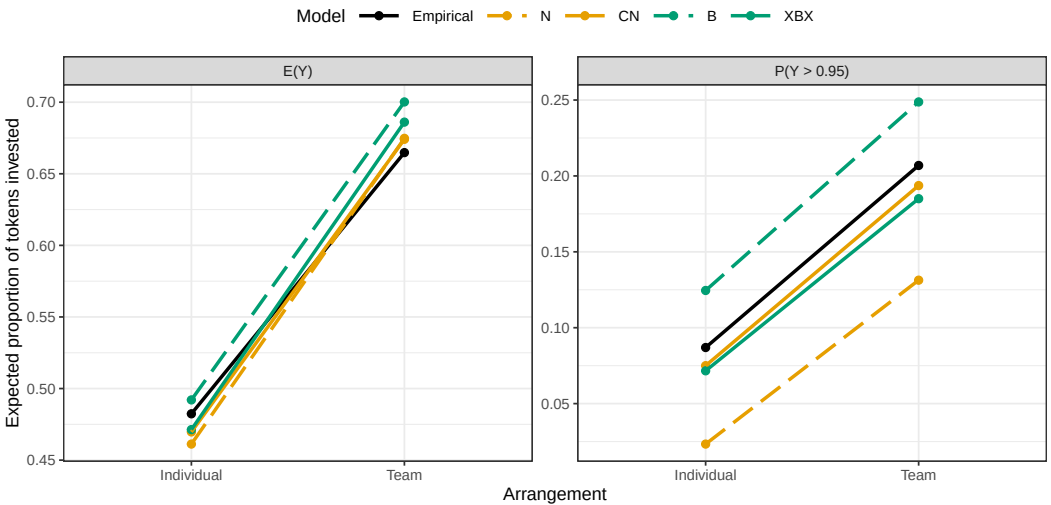
```
R> la_xbx <- betareg(invest ~ grade * (arrangement + age) + male |
+    arrangement + male + grade, data = LossAversion)
```

etc.

Loss aversion



Loss aversion



Theory

More formally: XB(μ, ϕ, u) is a beta distribution B(μ, ϕ) with support extended to $(-u, u)$ and censored at 0 and 1.

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$$f_{(\text{XB})}(y \mid \mu, \phi, u) \quad = \quad f_{(\text{B})}\left(\frac{y+u}{1+2u} \,\middle|\, \mu, \phi\right) \frac{1}{1+2u}, \quad \text{if } y \in (0,1)$$

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Special cases: Beta ($u = 0$) and censored normal ($u \to \infty$) distributions.

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Shrinkage: XBX(μ, ϕ, ν) is a continuous mixture of XB(μ, ϕ, u) with $u \sim$ Exp(ν).

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$$f_{(\text{XBX})}(y \mid \mu, \phi, \nu) \quad = \quad \nu^{-1} \int_0^{\infty} f_{(\text{XB})}(y \mid \mu, \phi, u)\, e^{-u/\nu} du$$

References

Cribari-Neto F, Zeileis A (2010). “Beta Regression in R.” *Journal of Statistical Software*, **34**(2), 1–24. doi:10.18637/jss.v034.i02

Messner JW, Mayr GJ, Zeileis A (2016). “Heteroscedastic Censored and Truncated Regression with crch.” *The R Journal*, **8**(1), 173–181. doi:10.32614/RJ-2016-012

Kosmidis I, Zeileis A (2024). “Extended-Support Beta Regression for $[0, 1]$ Responses.” *arXiv.org E-Print Archive*, arXiv:2409.07233. doi:10.48550/arXiv.2409.07233

Software:
https://topmodels.R-Forge.R-project.org/crch/
https://topmodels.R-Forge.R-project.org/betareg/

Mastodon: @zeileis@fosstodon.org

Bluesky: @zeileis.org

Web: https://www.zeileis.org/