Parties, Models, Mobsters

A New Implementation of Model-Based Recursive Partitioning in R

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Overview

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- Model-based (MOB) recursive partitioning
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  - Local models
- Implementation in R
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  - All new implementation in *partykit*
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Motivation: Trees

Breiman (2001, *Statistical Science*) distinguishes two cultures of statistical modeling.

- **Data models:** Stochastic models, typically parametric.
  $\rightarrow$ Classical strategy in statistics. Regression models are still the workhorse for many empirical analyses.
- **Algorithmic models:** Flexible models, data-generating process unknown. $\rightarrow$ Still few applications in many fields, e.g., social sciences or economics.

Classical example: Trees, i.e., modeling of dependent variable y by "learning" a recursive partition w.r.t explanatory variables $z_1, \dots, z_l$.

Motivation: Leaves

Key features:

1. Predictive power in nonlinear regression relationships.
2. Interpretability (enhanced by visualization), i.e., no “black box” methods.

Typically: Simple models for univeriate y, e.g., mean.

Idea: More complex models for more complex y, e.g., regression models, multivariate normal model, item responses, etc.

Here: Synthesis of parametric data models and algorithmic tree models.

Goal: Fitting local models by partitioning of the sample space.

Recursive partitioning

Model-based (MOB) algorithm:

1. Fit the parametric model in the current subsample.
2. Assess the stability of the parameters across each partitioning variable z_j.
3. Split sample along the z_{j^*} with strongest instability: Choose breakpoint with highest improvement of the model fit.
4. Repeat steps 1–3 recursively in the subsamples until some stopping criterion is met.

Recursive partitioning

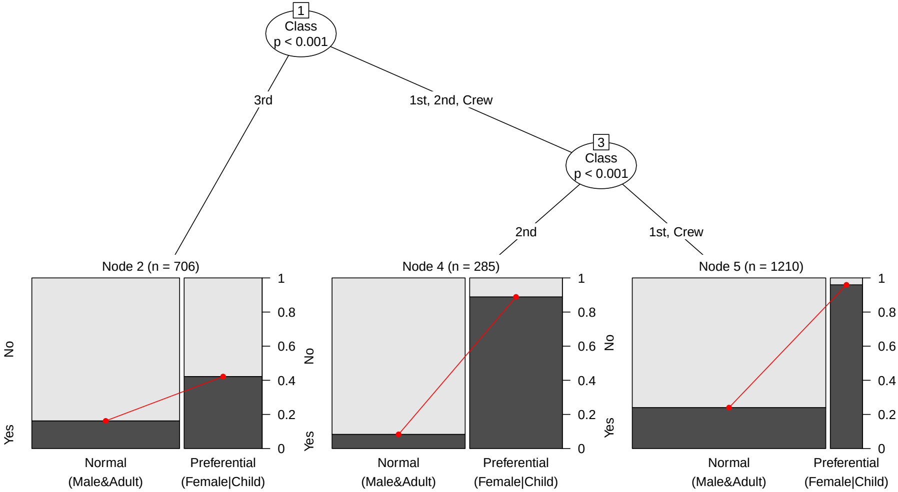
Example: Logistic regression, assessing differences in the effect of “preferential treatment” (“women and children first”?) in the Titanic survival data.

In R: Generalized linear model tree with binomial family (and default logit link).

```
R> mb <- glmtree(Survived ~ Treatment | Age + Gender + Class,
+    data = ttnc, family = binomial, alpha = 0.05, prune = "BIC")
R> plot(mb)
R> print(mb)
```

Result: Log-odds ratio of survival given treatment differs across classes (slope), as does the survival probability of male adults (intercept).

Recursive partitioning



Recursive partitioning

```
Generalized linear model tree (family: binomial)

Model formula:
Survived ~ Treatment | Age + Gender + Class

Fitted party:
[1] root
|   [2] Class in 3rd: n = 706
|                 (Intercept) TreatmentPreferential
|                      -1.641                 1.327
|   [3] Class in 1st, 2nd, Crew
|   |   [4] Class in 2nd: n = 285
|   |                 (Intercept) TreatmentPreferential
|   |                      -2.398                 4.477
|   |   [5] Class in 1st, Crew: n = 1210
|   |                 (Intercept) TreatmentPreferential
|   |                      -1.152                 4.318

Number of inner nodes:    2
Number of terminal nodes: 3
Number of parameters per node: 2
Objective function (negative log-likelihood): 1061
```

1. Model estimation

Models: $\mathcal{M}(y, x, \theta)$ with (potentially multivariate) observations y, optionally regressors x, and k-dimensional parameter vector $\theta \in \Theta$.

Parameter estimation: $\widehat{\theta}$ by optimization of additive objective function $\Psi(y, x, \theta)$ for n observations y_i ($i = 1, \dots, n$):

$$\widehat{\theta} \quad = \quad \underset{\theta \in \Theta}{\text{argmin}} \sum_{i=1}^{n} \Psi(y_i, x_i, \theta).$$

Special cases: Maximum likelihood (ML), weighted and ordinary least squares (OLS and WLS), quasi-ML, and other M-estimators.

1. Model estimation

Estimating function: $\widehat{\theta}$ can also be defined in terms of

$$\sum_{i=1}^{n} \psi(y_i, x_i, \widehat{\theta}) = 0,$$

where $\psi(y, x, \theta) = \partial \Psi(y, x, \theta)/\partial \theta$.

Central limit theorem: If there is a true parameter θ_0 and given certain weak regularity conditions:

$$\sqrt{n}(\widehat{\theta} - \theta_0) \quad \xrightarrow{d} \quad \mathcal{N}(0, V(\theta_0)),$$

where $V(\theta_0) = \{A(\theta_0)\}^{-1} B(\theta_0) \{A(\theta_0)\}^{-1}$. A and B are the expectation of the derivative of ψ and the variance of ψ, respectively.

1. Model estimation

Idea: In many situations, a single global model $\mathcal{M}(y, x, \theta)$ that fits **all** n observations cannot be found. But it might be possible to find a partition w.r.t. the variables $z_1, \dots, z_l$ so that a well-fitting model can be found locally in each cell of the partition.

Tools:

- Assess parameter instability w.r.t to partitioning variables z_j $(j = 1, \dots, l)$.
- A general measure of deviation from the model is the estimating function $\psi(y, x, \theta)$.

2. Tests for parameter instability

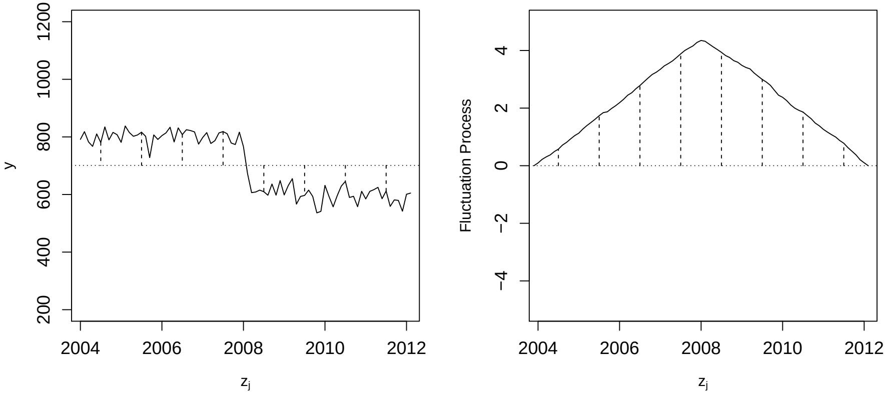
Generalized M-fluctuation tests capture instabilities in $\widehat{\theta}$ for an ordering w.r.t z_j.

Basis: Empirical fluctuation process of cumulative deviations w.r.t. to an ordering $\sigma(z_{ij})$.

$$W_j(t, \widehat{\theta}) \quad = \quad \widehat{B}^{-1/2} n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} \psi(y_{\sigma(z_{ij})}, x_{\sigma(z_{ij})}, \widehat{\theta}) \qquad (0 \leq t \leq 1)$$

Functional central limit theorem: Under parameter stability $W_j(\cdot) \xrightarrow{\text{d}} W^0(\cdot)$, where W^0 is a k-dimensional Brownian bridge.

2. Tests for parameter instability



2. Tests for parameter instability

Test statistics: Scalar functional $\lambda(W_j)$ that captures deviations from zero.

Null distribution: Asymptotic distribution of $\lambda(W^0)$.

Special cases: Class of test encompasses many well-known tests for different classes of models. Certain functionals λ are particularly intuitive for numeric and categorical z_j, respectively.

Advantage: Model $\mathcal{M}(y, x, \widehat{\theta})$ just has to be estimated once. Empirical estimating functions $\psi(y_i, x_i, \widehat{\theta})$ just have to be re-ordered and aggregated for each z_j.

2. Tests for parameter instability

Splitting numeric variables: Assess instability using sup*LM* statistics.

$$\lambda_{supLM}(W_j) \quad = \quad \max_{i=\underline{\imath},\dots,\overline{\imath}} \left(\frac{i}{n} \cdot \frac{n-i}{n} \right)^{-1} \left\| W_j \left(\frac{i}{n} \right) \right\|_2^2 .$$

Interpretation: Maximization of single shift *LM* statistics for all conceivable breakpoints in $[\underline{\imath}, \overline{\imath}]$.

Limiting distribution: Supremum of a squared, *k*-dimensional tied-down Bessel process.

Potential alternatives: Many other parameter instability tests from the same class of tests, e.g., a Cramér-von Mises test (or Nyblom-Hansen test), MOSUM tests, etc.

2. Tests for parameter instability

Splitting categorical variables: Assess instability using χ^2 statistics.

$$\lambda_{\chi^2}(W_j) \quad = \quad \sum_{c=1}^{C} \frac{n}{|I_c|} \left\| \Delta_{I_c} W_j \left(\frac{i}{n} \right) \right\|_2^2 .$$

Feature: Invariant for re-ordering of the C categories and the observations within each category.

Interpretation: Capture instability for split-up into C categories.

Limiting distribution: χ^2 with $k \cdot (C - 1)$ degrees of freedom.

2. Tests for parameter instability

Splitting ordinal variables: Several strategies conceivable. Assess instability either as for categorical variables (if C is low), or as for numeric variables (if C is high), or via a specialized test.

$$\begin{aligned}
\lambda_{maxLMo}(W_j) &= \max_{i \in \{i_1, \dots, i_{C-1}\}} \left(\frac{i}{n} \cdot \frac{n-i}{n} \right)^{-1} \left\| W_j\left(\frac{i}{n}\right) \right\|_2^2, \\
\lambda_{WDMo}(W_j) &= \max_{i \in \{i_1, \dots, i_{C-1}\}} \left(\frac{i}{n} \cdot \frac{n-i}{n} \right)^{-1/2} \left\| W_j\left(\frac{i}{n}\right) \right\|_\infty.
\end{aligned}$$

Interpretation: Assess only the possible splitpoints $i_1, \dots, i_{C-1}$, based on L_2 or L_∞ norm.

Limiting distribution: Maximum from selected points in a squared Bessel process or multivariate normal distribution, respectively.

3. Segmentation

Goal: Split model into $b = 1, \dots, B$ segments along the partitioning variable z_j associated with the highest parameter instability. Local optimization of

$$\sum_b \sum_{i \in I_b} \Psi(y_i, x_i, \theta_b).$$

$B = 2$: Exhaustive search of order $O(n)$.

$B > 2$: Exhaustive search is of order $O(n^{B-1})$, but can be replaced by dynamic programming of order $O(n^2)$. Different methods (e.g., information criteria) can choose B adaptively.

Here: Binary partitioning. Optionally, $B = C$ can be chosen (without search) for categorical variables.

4. Pruning

Goal: Avoid overfitting.

Pre-pruning:

- Internal stopping criterium.
- Stop splitting when there is no significant parameter instability.
- Based on Bonferroni-corrected p values of the fluctuation tests.

Post-pruning:

- Grow large tree (e.g., with high significance level).
- Prune splits that do not improve the model fit based on information criteria (e.g., AIC or BIC).

Hyperparameters: Significance level and information criterion penalty can be chosen manually (or possibly through cross-validation etc.).

Local models

Goals:

- Detection of interactions and nonlinearities in regressions.
- Add explanatory variables for models without regressors.
- Detect violations of parameter stability (measurement invariance) across several variables adaptively.

Mobsters:

- Linear and generalized linear model trees (Zeileis *et al.* 2008).
- Censored survival regression trees: parametric proportional hazard and accelerated failure time models (Zeileis *et al.* 2008).
- Beta regression trees (Grün *et al.* 2012).
- Bradley-Terry trees for paired comparisons (Strobl *et al.* 2011).
- Item response theory (IRT) trees: Rasch, rating scale and partial credit model (Strobl *et al.* 2014, Abou El-Komboz *et al.* 2014).

Implementation: Building blocks

Workhorse function: `mob()` for

- data handling,
- calling model fitters,
- carrying out parameter instability tests and
- recursive partitioning algorithm.

Required functionality:

- *Parties:* Class and methods for recursive partytions.
- *Models:* Fitting functions for statistical models (optimizing suitable objective function).
- *Mobsters:* High-level interfaces (`lmtree()`, `bttree()`, `...`) that call lower-level `mob()` with suitable options and methods.

Implementation: Old `mob()` in *party*

Parties: S4 class ‘`BinaryTree`’.

- Originally developed only for `ctree()` and somewhat “abused”.
- Rather rigid and hard to extend.

Models: S4 ‘`StatModel`’ objects.

- Intended to conceptualize unfitted model objects.
- Required some “glue code” to accomodate non-standard interface for data handling and model fitting.

Mobsters:

- `mob()` already geared towards (generalized) linear models.
- Other interfaces in *psychotree* and *betareg*.
- Hard to do fine control due to adopted S4 classes: Many unnecessary computations and copies of data.

Implementation: New `mob()` in *partykit*

Parties: S3 class '`modelparty`' built on '`party`'.

- Separates data and tree structure.
- Inherits generic infrastructure for printing, predicting, plotting, …

Models: Plain functions with input/output convention.

- Basic and extended interface for rapid prototyping and for speeding up computings, respectively.
- Only minimial glue code required if models are well-designed.

Mobsters:

- `mob()` completely agnostic regarding models employed.
- Separate interfaces `lmtree()`, `glmtree()`, …
- New interfaces typically need to bring their model fitter and adapt the main methods `print()`, `plot()`, `predict()` etc.

Implementation: New `mob()` in *partykit*

New inference options: Not used by default by optionally available.

- New parameter instability tests for ordinal partitioning variables. Alternative to unordered χ^2 test but computationally intensive.
- Post-pruning based on information criteria (e.g., AIC or BIC), especially for very large datasets where traditional significance levels are not useful.
- Multiway splits for categorical partitioning variables.
- Treat weights as proportionality weights and not as case weights.

Implementation: Models

Input: Basic interface.

```
fit(y, x = NULL, start = NULL, weights = NULL,
  offset = NULL, ...)
```

`y`, `x`, `weights`, `offset` are (the subset of) the preprocessed data. Starting values and further fitting arguments are in `start` and `...`.

Output: Fitted model object of class with suitable methods.

- `coef()`: Estimated parameters $\hat{\theta}$.
- `logLik()`: Maximized log-likelihood function $-\sum_i \Psi(y_i, x_i, \hat{\theta})$.
- `estfun()`: Empirical estimating functions $\Psi'(y_i, x_i, \hat{\theta})$.

Implementation: Models

Input: Extended interface.

```
fit(y, x = NULL, start = NULL, weights = NULL,
  offset = NULL, ..., estfun = FALSE, object = FALSE)
```

Output: List.

- `coefficients`: Estimated parameters $\hat{\theta}$.
- `objfun`: Minimized objective function $\sum_i \Psi(y_i, x_i, \hat{\theta})$.
- `estfun`: Empirical estimating functions $\Psi'(y_i, x_i, \hat{\theta})$. Only needed if `estfun = TRUE`, otherwise optionally `NULL`.
- `object`: A model object for which further methods could be available (e.g., `predict()`, or `fitted()`, etc.). Only needed if `object = TRUE`, otherwise optionally `NULL`.

Internally: Extended interface constructed from basic interface if supplied. Efficiency can be gained through extended approach.

Implementation: Parties

Class: '`modelparty`' inheriting from '`party`'.

Main addition: Data handling for regressor and partitioning variables.

- The *Formula* package is used for two-part formulas, e.g., `y ~ x1 + x2 | z1 + z2 + z3`.
- The corresponding terms are stored for the combined model and only for the partitioning variables.

Additional information: In `info` slots of '`party`' and '`partynode`'.

- `call`, `formula`, `Formula`, `terms` (partitioning variables only), `fit`, `control`, `dots`, `nreg`.
- `coefficients`, `objfun`, `object`, `nobs`, `p.value`, `test`.

Reusability: Could in principle be used for other model trees as well (inferred by other algorithms than MOB).

Bradley-Terry trees





Questions: Which of these women is more attractive?
How does the answer depend on age, gender, and the familiarity with the associated TV show Germany's Next Topmodel?

Bradley-Terry trees

Task: Preference scaling of attractiveness.

Data: Paired comparisons of attractiveness.

- *Germany's Next Topmodel 2007* finalists: Barbara, Anni, Hana, Fiona, Mandy, Anja.
- Survey with 192 respondents at Universität Tübingen.
- Available covariates: Gender, age, familiarty with the TV show.
- Familiarity assessed by yes/no questions: (1) Do you recognize the women?/Do you know the show? (2) Did you watch it regularly? (3) Did you watch the final show?/Do you know who won?

Bradley-Terry trees

Model: Bradley-Terry (or Bradley-Terry-Luce) model.

- Standard model for paired comparisons in social sciences.
- Parametrizes probability π_{ij} for preferring object i over j in terms of corresponding "ability" or "worth" parameters θ_i.

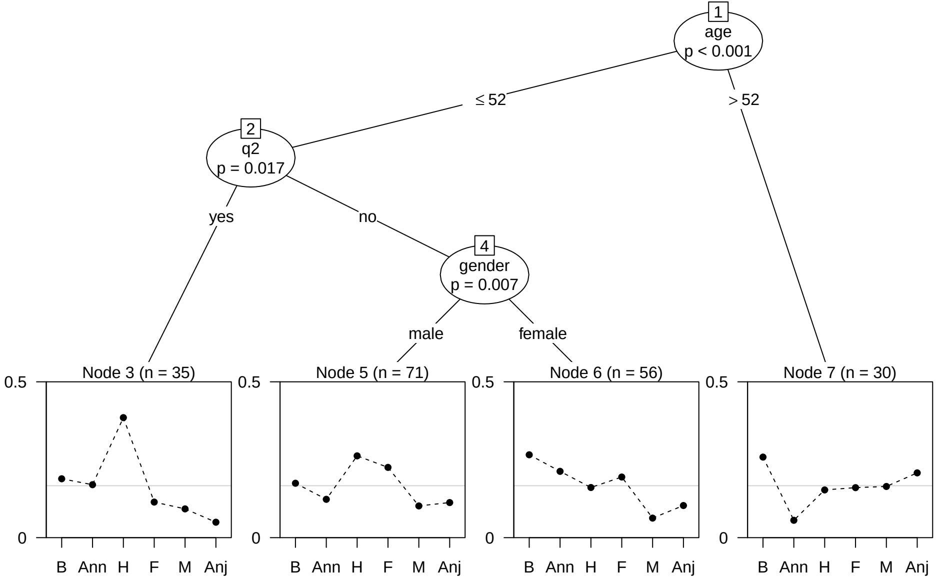
$$
\begin{aligned}
\pi_{ij} &= \frac{\theta_i}{\theta_i + \theta_j} \\
\text{logit}(\pi_{ij}) &= \log(\theta_i) - \log(\theta_j)
\end{aligned}
$$

- Maximum likelihood as a logistic or log-linear GLM.

Mobster: `bttree()` in *psychotree* (Strobl *et al.* 2011).

Here: Use `mob()` directly to build model from scratch using `btReg.fit()` from *psychotools*.

Bradley-Terry trees



Bradley-Terry trees

Data, packages, and `estfun()` method:

```
R> data("Topmodel2007", package = "psychotree")
R> library("partykit")
R> library("psychotools")
R> estfun.btReg <- function(x, ...) x$estfun
```

Basic model fitting function:

```
R> btfit1 <- function(y, x = NULL, start = NULL, weights = NULL,
+    offset = NULL, ...) btReg.fit(y, weights = weights, ...)
```

Fit Bradley-Terry tree:

```
R> system.time(bt1 <- mob(
+    preference ~ 1 | gender + age + q1 + q2 + q3,
+    data = Topmodel2007, fit = btfit1))
```

```
   user  system elapsed
  5.148   0.036   5.215
```

Bradley-Terry trees

Extended model fitting function:

```
R> btfit2 <- function(y, x = NULL, start = NULL, weights = NULL,
+    offset = NULL, ..., estfun = FALSE, object = FALSE) {
+    rval <- btReg.fit(y, weights = weights, ...,
+      estfun = estfun, vcov = object)
+    list(
+      coefficients = rval$coefficients,
+      objfun = -rval$loglik,
+      estfun = if(estfun) rval$estfun else NULL,
+      object = if(object) rval else NULL
+    )
+  }
```

Fit Bradley-Terry tree again:

```
R> system.time(bt2 <- mob(
+    preference ~ 1 | gender + age + q1 + q2 + q3,
+    data = Topmodel2007, fit = btfit2))

   user  system elapsed
  4.132   0.000   4.142
```

Bradley-Terry trees

```
Model-based recursive partitioning (btfit2)

Model formula:
preference ~ 1 | gender + age + q1 + q2 + q3

Fitted party:
[1] root
|   [2] age <= 52
|   |   [3] q2 in yes: n = 35
|   |       Barbara    Anni    Hana   Fiona   Mandy
|   |        1.3378  1.2318  2.0499  0.8339  0.6217
|   |   [4] q2 in no
|   |   |   [5] gender in male: n = 71
|   |   |        Barbara     Anni     Hana    Fiona    Mandy
|   |   |        0.43866  0.08877  0.84629  0.69424 -0.10003
|   |   |   [6] gender in female: n = 56
|   |   |       Barbara    Anni    Hana   Fiona   Mandy
|   |   |        0.9475  0.7246  0.4452  0.6350 -0.4965
|   [7] age > 52: n = 30
|       Barbara    Anni    Hana   Fiona   Mandy
|        0.2178 -1.3166 -0.3059 -0.2591 -0.2357
```

Bradley-Terry trees

```
Number of inner nodes:    3
Number of terminal nodes: 4
Number of parameters per node: 5
Objective function: 1829
```

Standard methods readily available:

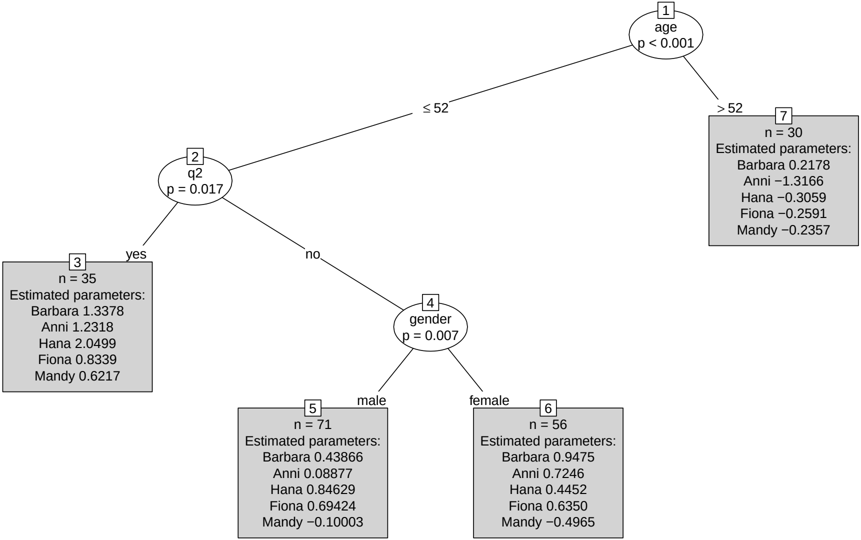
```
R> plot(bt2)
R> coef(bt2)
```

```
  Barbara     Anni    Hana   Fiona   Mandy
3  1.3378  1.23183  2.0499  0.8339  0.6217
5  0.4387  0.08877  0.8463  0.6942 -0.1000
6  0.9475  0.72459  0.4452  0.6350 -0.4965
7  0.2178 -1.31663 -0.3059 -0.2591 -0.2357
```

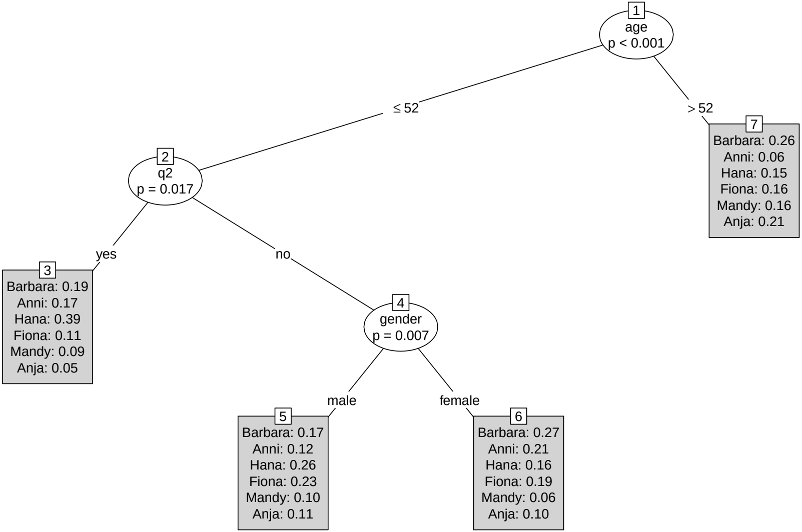
Customization:

```
R> worthf <- function(info) paste(info$object$labels,
+    format(round(worth(info$object), digits = 2)), sep = ": ")
R> plot(bt2, FUN = worthf)
```

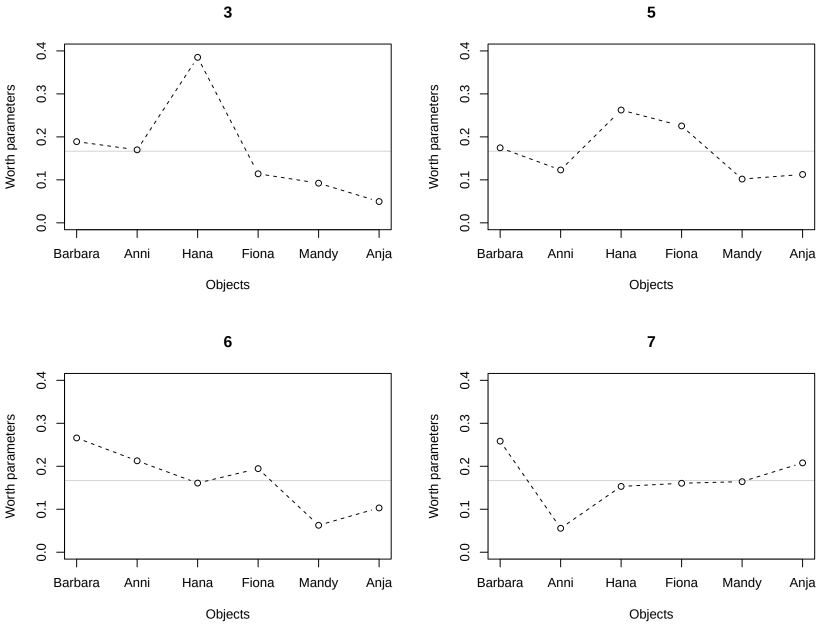
Bradley-Terry trees



Bradley-Terry trees



Bradley-Terry trees



Bradley-Terry trees

Apply plotting in all terminal nodes:

```
R> par(mfrow = c(2, 2))
R> nodeapply(bt2, ids = c(3, 5, 6, 7), FUN = function(n)
+    plot(n$info$object, main = n$id, ylim = c(0, 0.4)))
```

Predicted nodes and ranking:

```
R> tm

  age gender q1  q2 q3
1  60   male no  no no
2  25 female no  no no
3  35 female no yes no

R> predict(bt2, tm, type = "node")

1 2 3
7 3 5

R> predict(bt2, tm, type = function(object) t(rank(-worth(object))))

  Barbara Anni Hana Fiona Mandy Anja
1       1    6    5     4     3    2
2       2    3    1     4     5    6
3       3    4    1     2     6    5
```

Summary

- Synthesis of parametric data models and algorithmic tree models.
- Based on modern class of parameter instability tests.
- Aims to minimize clearly defined objective function by greedy forward search.
- Can be applied general class of parametric models.
- Alternative to traditional means of model specification, especially for variables with unknown association.
- All new implementation in *partykit*.
- Enables more efficient computations, rapid prototyping, flexible customization.

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Software:

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